Math 129 - 011

Problem Session Notes

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I owe you a more legable treatment of the problem we did today in class. So, we want to compute the antiderivative of the function $5e^{4x}\sin 2x$. So we have

$$\int 5e^{4x}\sin 2x\ dx$$

and we notice that we have two functions being multipled. We could use u-substitution if one of those functions was the derivative of some inside function. Possible inside functions are $4x$ and $2x$, niether of the functions e^{4x} or $\sin 2x$ are derivatives of these. So, we conclude this is not a integral for u-substitution.

The next thing to try is integration by parts. The question now is which is the u part and which is the v' part? We always want to pick v' to be something easy to integrate, and the easiest thing to integrate is the exponential. So, let's try that.[1]

$$\left|\begin{array}{ll} v' = e^{4x} & v = \frac{1}{4}e^{4x} \\ u = \sin 2x & u' = 2\cos 2x \end{array}\right.$$

The 5 will just come along for the ride.

$$\text{Integration by parts: } \int u(x)v'(x)\ dx = u(x)v(x) - \int u'(x)v(x)\ dx \tag{1}$$

$$\begin{aligned} \int 5e^{4x}\sin 2x\ dx &= 5\left(\frac{1}{4}e^{4x}\sin 2x - \int \frac{1}{4}e^{4x}2\cos 2x\ dx\right) &(2)\\ &= \frac{5}{4}e^{4x}\sin 2x - \frac{5}{2}\int e^{4x}\cos 2x\ dx &(3)\end{aligned}$$

Now, for the second integral, it looks like I need to do integration by parts again. This time

$$\left|\begin{array}{ll} v' = e^{4x} & v = \frac{1}{4}e^{4x} \\ u = \cos 2x & u' = -2\sin 2x \end{array}\right.$$

[1] To find $v(x)$ we are really just looking for

$$v(x) = \int v'(x)\ dx = \int e^{4x}\ dx$$

this is a problem in itself (albeit a small one) for which we can use u-substitution. Let $u = 4x$. Then $du = 4dx$ and $dx = \frac{1}{4}du$ so that

$$v(x) = \int v'(x)\ dx = \int e^{4x}\ dx = \frac{1}{4}\int e^u\ du = \frac{1}{4}e^{4x}$$

Now, what about the "$+C$" here? Well watch this: let's do the intergration by parts formula using a general anti-derivative so in place of "v" we use "$v + C$"

$$\begin{aligned} \int u(x)v'(x) &= u(x)(v(x)+C) - \int u'(x)(v(x)+C)\ dx \\ &= u(x)v(x) + Cu(x) - \int u'(x)v(x) + u'(x)C\ dx \\ &= u(x)v(x) + Cu(x) - \int u'(x)v(x)\ dx - C\int u'(x)\ dx \qquad \text{it's just a constant} \\ &= u(x)v(x) + Cu(x) - \int u'(x)v(x)\ dx - Cu(x) \\ &= u(x)v(x) - \int u'(x)v(x)\ dx \end{aligned}$$

So, ANY anti-derivative will work. And so, we just choose the one with $C = 0$.

So that,

$$\int 5e^{4x}\sin 2x\;dx = \frac{5}{4}e^{4x}\sin 2x - \frac{5}{2}\int e^{4x}\cos 2x\;dx \tag{4}$$

$$= \frac{5}{4}e^{4x}\sin 2x - \frac{5}{2}\left(\frac{1}{4}e^{4x}\cos 2x - \int \frac{1}{4}e^{4x}(-2\sin 2x)\;dx\right) \tag{5}$$

$$= \frac{5}{4}e^{4x}\sin 2x - \frac{5}{8}e^{4x}\cos 2x - \frac{5}{4}\int e^{4x}\sin 2x\;dx \tag{6}$$

$$= \frac{5}{4}e^{4x}\sin 2x - \frac{5}{8}e^{4x}\cos 2x - \frac{1}{4}\int 5e^{4x}\sin 2x\;dx \tag{7}$$

Notice that this time I tucked the 5 into the integral (something I think Jessica would approve of based on her comments) which I think is nicer. We end up with the equation

$$\underline{\int 5e^{4x}\sin 2x\;dx} = \frac{5}{4}e^{4x}\sin 2x - \frac{5}{8}e^{4x}\cos 2x - \frac{1}{4}\underline{\int 5e^{4x}\sin 2x\;dx}$$

Now let's not use the I, as I think Zach suggested we don't, and observe that the underlined bits are exactly the integral that we are trying to calculate. So, lets take equation and solve for that integral, just as though it was some variable. This gives[2]

$$\left(1+\frac{1}{4}\right)\int 5e^{4x}\sin 2x\;dx = \frac{5}{4}e^{4x}\sin 2x - \frac{5}{8}e^{4x}\cos 2x + C$$

and

$$\int 5e^{4x}\sin 2x\;dx = \frac{4}{5}\left(\frac{5}{4}e^{4x}\sin 2x - \frac{5}{8}e^{4x}\cos 2x\right) + C \tag{10}$$

$$= e^{4x}\sin 2x - \frac{1}{2}e^{4x}\cos 2x + C \tag{11}$$

Lets check:

$$\begin{aligned}\frac{d}{dt}\left(e^{4x}\sin 2x - \frac{1}{2}e^{4x}\cos 2x + C\right) &= 4e^{4x}\sin 2x + 2e^{4x}\cos 2x - \frac{1}{2}\left(4e^{4x}\cos 2x - 2e^{4x}\sin 2x\right) + 0\\ &= 4e^{4x}\sin 2x + 2e^{4x}\cos 2x - 2e^{4x}\cos 2x + e^{4x}\sin 2x\\ &= 4e^{4x}\sin 2x + e^{4x}\sin 2x\\ &= 5e^{4x}\sin 2x\end{aligned}$$

Wow! It worked.

[2]A word about the "$+C$" in this case generally when we use integration by parts we don't include a "$+C$" until we have evaluated the last integral there happens to be because that indefinite integral has a "$+C$" of it's own and there for takes care of the need for any other "$+C$". In this case we use the final integral to combine with the first and seem to have missed the opportunity to add in the "$+C$". We therfore must put it when we have a single integral wich is only on one side. You see, equality can not hold with out that "$+C$" on the other side. while we have an indefinite integral on both sides equality holds because each has a "$+C$" built into it.

Now is a great time for a proof that $0 = 1$. Lets try to evaluate $\int \frac{1}{x}dx$ using integration by parts (of course we already know this equals $\ln|x| + C$, but that doesn't mean we can't use integration by parts). Lets have $u = \frac{1}{x}$ and $v' = 1$

$$\left|\begin{array}{ll} v' = 1 & v = x \\ u = 1/x & u' = -1/x^2 \end{array}\right.$$

This gives,

$$\int \frac{1}{x}\;dx = \frac{1}{x}x - \int x\frac{-1}{x^2}\;dx \tag{8}$$

$$= 1 + \int \frac{1}{x}\;dx \tag{9}$$

This begs the question how can something be equal to one more then itself? Subtracting $\int \frac{1}{x}dx$ from both side reveals the conclusion that $0 = 1$. Now, how is this invalid?