Worksheet 9

1. The vector field $\vec{F}$ has $\|\vec{F}\| \le 15$ everywhere and C is the circle of radius 3 centered at the origin with clockwise direction. What are the largest and smallest possible values of the line integral $\int_C \vec{F} \cdot d\vec{r}$? Give an example of corresponding vector field $\vec{F}$ for each of the cases.

2. Evaluate the following line integrals. For each problem, try to come up with the most efficient strategy for computing the integral.

   (a) The line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = x^2\,\vec{i} + 4y\,\vec{j}$, and C is the part of the curve $y = 4/2^x$ from $(0,4)$ to $(2,1)$.

   (b) The line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = 3y\,\vec{i} - 4x\,\vec{j}$, and C is a semicircle of radius 2 centered at the origin, from $(0,-2)$ to $(2,0)$ to $(0,2)$.

   (c) The line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where

   $$\vec{F}(x,y,z) = \frac{2x}{x^2+y^2+z^2}\,\vec{i} + \frac{2y}{x^2+y^2+z^2}\,\vec{j} + \frac{2z}{x^2+y^2+z^2}\,\vec{k},$$

   and C is the upward spiral around the z-axis that begins at $(1,0,0)$ and ends at $(1,0,4)$, wrapping around the z-axis twice.

3. Let $\vec{F}(x,y) = (4-2x)\vec{i} + (12-4y)\vec{j}$. Suppose that C is a semicircular path starting at the origin in $\mathbb{R}^2$. What is the maximum possible value of $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, and what can be a corresponding path C in that case? Justify your answer.

4. (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,

   $$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

   for some constant c where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving an object from point P_1 along a path to point P_2 in terms of the distance d_1 and d_2 from these points to the origin. [Hint: What is $\mathbf{F}$ is terms of x, y, z can you find a potential?]

   (b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the Sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ km, $M = 1.99 \times 10^{30}$ km, and $G = 10.67 \times 10^{-11}$ N·m^2/kg^2.)

   (c) Another example of an inverse square field is the electric field $\mathbf{E} = \epsilon qQ\mathbf{r}/|\mathbf{r}|^3$. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned at a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric field. (Use the value $\epsilon = 8.985 \times 10^{10}$.)

5. Let $\mathbf{F}(x, y) = \left(3 \ln \left[x^2 + y^2\right] + \dfrac{6x^2}{x^2 + y^2}\right) \mathbf{i} + \dfrac{6xy}{x^2 + y^2} \mathbf{j}$

   (a) *Show* that $\partial F_1/\partial y = \partial F_2/\partial x$

   (b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. Does this contradict the curl test for path independence?
   [Hint: Find $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ and where C_1 and C_2 both start at $(1, 0)$ and end at $(-1, 0)$ where C_1 is the upper half of the unit circle and C_2 is the lower half of the unit circle. Are these integrals equal?]

6. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\mathbf{i} + xy^2\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.