Worksheet 7

1. (1) For each of the following triple integrals, draw the region in $\mathbb{R}^3$ over which the integral is being evaluated. Do not evaluate the integrals.

   (a) $\displaystyle\int_0^3 \int_{-2}^4 \int_0^{\sqrt{9-x^2}} y^2 \, dz \, dy \, dx$

   (b) $\displaystyle\int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{9-\sqrt{x^2+y^2}} z^2 \, dz \, dx \, dy$

   (c) $\displaystyle\int_{-2}^2 \int_0^{2-y} \int_0^{5-z} (x^2+y^2) \, dx \, dz \, dy$

   (2) A solid pyramid T in $\mathbb{R}^3$ has vertices at $(0,0,0)$, $(4,0,0)$, $(4,4,0)$, $(0,4,0)$, and $(0,0,2)$. Let $f(x,y,z)$ be a continuous function on T. Set up an integral, complete with an order of integration and limits of integration for each variable, for $f(x,y,z)$ over the solid region T.

2. (1) Evaluate each of the following integrals.

   (a) The integral of the function $f(x,y) = x$ on the bounded region in the plane to the right of the line $x = 1$ but inside the circle $x^2 + y^2 = 4$.

   (b) The integral of the function $g(x,y) = x^2 + y^2$ on the triangle in the coordinate plane whose vertices are $(0,0)$, $(1,0)$, and $(1,1)$.

   (2) Let R be the bounded region between the curve $y = \sqrt{x}$, the y-axis, and the line $y = 2$. Compute the following integrals:

   (a) $\displaystyle\int_R \sin(y^3) \, dA$

   (b) $\displaystyle\int_R \frac{2y}{2+\sqrt{x}} \, dA$

3. Decide whether each of the following statements is true or false. Give reasons for your answers.

   (a) If R is the region bounded by $x = 1, y = 0, y = x$, then in polar coordinates

   $$\int_R x dA = \int_0^{\frac{\pi}{4}} \int_0^1 r^2 \cos\theta \, dr \, d\theta.$$

   (b) The integral $\int_0^{2\pi} \int_0^{\pi} \int_0^1 1 \, d\rho \, d\phi \, d\theta$ gives the volume inside a sphere of radius 1.

   (c) Changing the order of the integration gives

   $$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\cos\phi} \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho.$$

   (d) The region of integration for $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{1}{\sin\theta}}^{\frac{4}{\sin\theta}} f(r,\theta) \, dr \, d\theta$ is a triangle.

4. Determine without computation whether the integrals below are positive, negative, or zero.

   (a) $\int_R ye^{-y}dA$, where R is the region inside the right half of the unit disc centered at the origin.

   (b) $\int_W \sin(2\phi)dV$, where W is the unit ball centered at the origin.

   (c) $\int_{W_2}(z^2 - z)dV$, where W_2 is the top half of the unit ball centered at the origin.

   (d) $\int_{W_2}(xz)dV$, where W_2 is the top half of the unit ball centered at the origin.

5. Below is a picture of a delicious bundt cake:[1]

The shape of this cake can be modeled by sketching the parabola $z = -y^2 + 6y - 5$ in the yz-plane (all lengths are in inches), and then revolving the bounded region between this parabola and the y-axis around the z-axis to create a three-dimensional solid. However, when the cake was baked, the ingredients in the cake did not settle uniformly. Let $\delta(x, y, z)$ represent the density of the cake (in pounds per cubic inch) at the point (x, y, z). Set up an integral, including limits of integration, for the total weight of the cake.

6. Evaluate the following integrals:

   (a) $$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z\sqrt{x^2+y^2}\,dz\,dy\,dx$$

   (b) $$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2}\,dz\,dy\,dx$$

[1]Image from allrecipes.com.