Worksheet 5

1. The tables below give values of the function $f(x, y)$ and its partial derivatives at a variety of points. Suppose that $x = -2uv$ and $y = u^2 - v^2$. If we think of f as a function of u and v, what are $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ at the point $(u, v) = (1, -1)$? How about at the point $(0, 1)$?

$f(x, y)$:

$y \backslash x$	-2	-1	0	1	2
2	-23	-18	-11	-2	10
1	-19	-16	-11	-6	1
0	-15	-14	-12	-10	-7
-1	-12	-12	-13	-13	-15
-2	-10	-11	-13	-16	-22

$f_x(x, y)$:

$y \backslash x$	-2	-1	0	1	2
2	6	7	8	10	12
1	3	4	5	6	7
0	1	2	2	3	3
-1	0	0	-1	-1	-2
-2	-1	-2	-3	-4	-6

$f_y(x, y)$:

$y \backslash x$	-2	-1	0	1	2
2	-4	-3	1	5	10
1	-4	-2	1	4	9
0	-3	-2	1	4	8
-1	-3	-1	0	3	7
-2	-2	-1	0	3	6

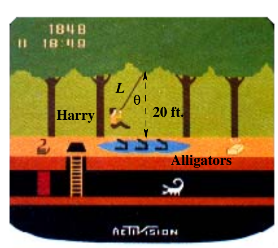
2. The temperature at a point (x, y) on a plate is $T(x, y)$, measured in degrees Celsius. A bug crawls on the plate so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters.

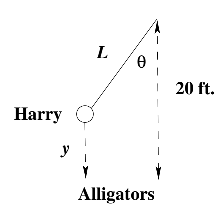
   (a) What does $T(t) = T(x(t), y(t))$ represent at a given time t?

   (b) Suppose The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bugs path after 3 seconds?

   (c) Suppose at 3 seconds the bug has had enough and stops, in what direction should the bug go for the heat to decrease the fastest.

3. In the Atari game *Pitfall!*, Pitfall Harry searches for treasure in a jungle. At certain points, Harry has to grab a vine hanging from a tree and swing over a pond full of alligators. (See the figure on the left. A more "abstract" version of the figure is shown on the right.)





The vine has length L (in feet) and swings from a point 20 feet over the pond. If t is the time in seconds from the moment Harry grabs the vine, the angle θ between the vine and the normal to the pond is given by

$$\theta(t) = \frac{\pi}{4}\cos(2t).$$

In addition, the vine is slightly elastic, and the vine stretches out at a rate of $dL/dt = 1/L$ at any given time. When $t = 0.5$ seconds, $L = 16$. At what rate is Harry getting closer to the surface of the pond at this time?

4. Cartesian coordinates (x, y) can be transformed to the polar coordinates (r, θ) and vice versa, via the relations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Let $z = z(x, y)$ be a differentiable function in the domain $x > 0, y > 0$.

(a) Use the chain rule to write $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(b) Using part (a), solve for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

(c) Show that the expressions you get in part (b) are the same as you would get by using the chain rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

(d) Show that the following relation holds:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

5. Answer the following questions about chain rules.

(a) Let $z(t) = f(t)g(t)$ for differentiable functions f and g. Use the chain rule applied to $h(x, y) = f(x)g(y)$ to show

$$\frac{dz}{dt} = f'(t)g(t) + f(t)g'(t).$$

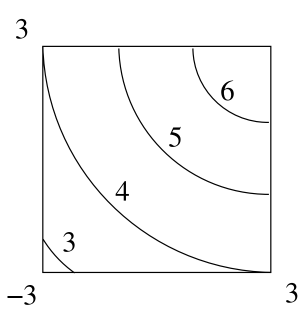
In other words, the one-variable product rule is a special case of the two-variable chain rule.

(b) A function $f(x, y)$ is said to be **homogeneous of degree** p if $f(tx, ty) = t^p f(x, y)$ for all t. Show that any differentiable function $f(x, y)$ which is homogeneous of degree p satisfies Euler's equation:

$$xf_x(x, y) + yf_y(x, y) = pf(x, y).$$

(Hint: for fixed (x, y), consider $g(t) = f(tx, ty)$ and compute $g'(1)$).

6. Below are contour diagrams for a function $f(x, y)$, defined on the region R consisting of all points (x, y) such that $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. The contours of the function are circular arcs centered at $(3, 3)$; the contours shown in each picture are spaced equally. Contours at non-integer heights progress linearly between the contours at integer heights. Answer the following questions:



(a) Give an underestimate and an overestimate for the value of $\int_R f(x, y)dA$.

(b) Decide whether the value of $\int_R xf(x, y)dA$ is positive, negative, or zero. Justify your answer.

(c) Let S be the subregion of R consisting of all points (x, y) such that $f(x, y) \geq 4$. Write a double integral that gives the average value of $f(x, y)$ on S.