Worksheet 4

1. Let $f(x, y) = x^2 + x + 3y^2$.

   (a) Draw the contour diagram for f. What are the shapes of the contour curves?

   (b) Evaluate $f(1, 3)$, $f_x(1, 3)$ and $f_y(1, 3)$.

   (c) Using (b), estimate $f(1.1, 2.9)$. Compare the estimate with the actual value. What is the error in percentage?

   (d) Using (b), determine the slope of the contour curve $f(x, y) = 29$ at $(x, y) = (1, 3)$.

2. Answer the following questions about partial differential equations.

   (a) Show that the Cobb-Douglas function

   $$Q = bK^{\alpha}L^{1-\alpha}, \quad 0 < \alpha < 1$$

   satisfies the equation

   $$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = Q.$$

   (b) Which of the following functions $f(x, y)$ satisfy the Euler's equation $xf_x + yf_y = f$?

      i. x^2y^3

      ii. $x + y + 1$

      iii. $x^2 + y^2$

      iv. $x^{0.4}y^{0.6}$

3. Let $f(x, y)$ and $g(x, y)$ be functions such that $\|\nabla f(x, y)\| = \|\nabla g(x, y)\|$ at a point P, where these gradients are not the zero vector. Show the following statements.

   (a) At P, the direction of the most rapid increase of $f + g$ increases f and g at equal rates.

   (b) At P, the direction of the most rapid increase of $f + g$ bisects the angle between the contours of f and g passing through P.

4. Consider the function $P(x, y) = x^2 - 4x + y^3 + 3y^2 + 2$.

   (a) Find an equation for the tangent plane to the graph of $P(x, y)$ at the point $(3, 1, 3)$.

   (b) Find a point on the graph of $P(x, y)$ where the tangent plane is parallel to the xy-plane.

   (c) Find a point on the graph of $P(x, y)$ where the tangent plane is parallel to (but not the same as) the plane you found in part (a).

5. The cardiac output, represented by c, is the volume of blood flowing through a person's heart per unit time. The systemic vascular resistance (SVR), represented by s, is the resistance to blood flowing through veins and arteries. Let p be a person's blood pressure. Then p is a function of c and s, so $p = f(s, c)$.

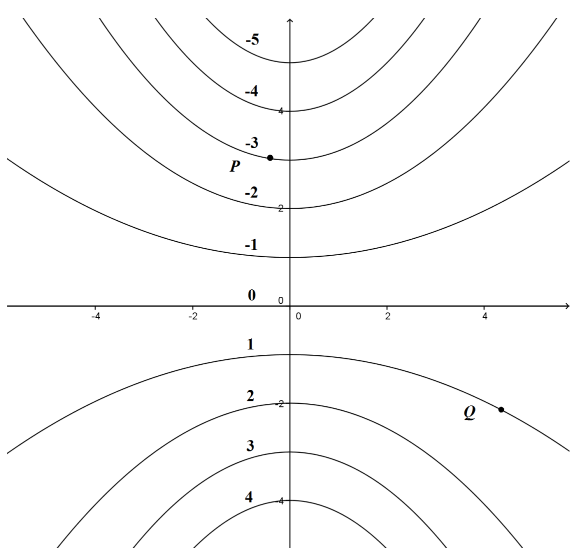
   (a) What do $\dfrac{\partial p}{\partial c}$ and $\dfrac{\partial p}{\partial s}$ represent?

   (b) Suppose from now that $p = kcs$ for some constant k. Sketch the level curves of p. What do they represent?

   (c) For a person with a weak heart, it is desirable to have the heart pumping against less resistance, while maintaining the same blood pressure. Such a person may be given the drug nitroglycerine to decrease the SVR and the drug Dopamine to increase the cardiac output. Represent this on a graph showing level curves. Put a point A on the graph representing the person's state before drugs are given and a point B for after.

   (d) Right after a heart attack, a patient's cardiac output drops, thereby causing the blood pressure to drop. A common mistake made by medical residents is to get the patient's blood pressure back to normal by using drugs to increase the SVR, rather than by increasing the cardiac output. On a graph of the level curves of p, put a point D representing the patient before the heart attack, a point E representing the patient right after the heart attack, and a third point F representing the patient after the resident has given the drugs to increase the SVR.

6. Shown below is a contour diagram for a function $f(x, y)$; as usual, the x-axis is horizontal, and the y-axis is vertical. All contours are parabolas, except for the $f = 0$ contour, which is the x-axis. Along any vertical line in the xy-plane, the intersections with the contour lines are equally spaced.



   (a) For each of the partial derivatives below, decide based on the picture whether the given partial derivative is positive, negative, or zero.

   $f_y(0, 0)$ $\qquad$ $f_{xx}(0, -3)$

   $f_{yy}(-1, 2)$ $\qquad$ $f_{yx}(3, 2)$

   (b) In the diagram above, two points P and Q are marked. In the picture, sketch the gradient vector $\overrightarrow{\nabla f}$ at each point. Don't worry about getting the exact lengths correct; but if one vector is longer than the other, your drawing should show this.

   (c) Let $\vec{u} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$. Find a point R in the diagram above such that $f_{\vec{u}}(R)$ is approximately zero, and mark the point in the picture.