Worksheet 13

1. Let S be the surface consisting of the following pieces:

   - The cylinder $x^2 + y^2 = 25$ from $z = 2$ to $z = 6$, oriented away from the z-axis
   - The disc $x^2 + y^2 \leq 25$ in the plane $z = 2$, oriented downward

   Let $\vec{F}(x, y, z) = -2y\,\vec{i} + 3xz\,\vec{j} + 5z\,\vec{k}$. Answer the following questions:

   (a) Compute the flux of $\vec{F}$ through S.

   (b) Compute the line integral of $\vec{F}$ over the circle $x^2 + y^2 = 25$ in the plane $z = 6$ oriented counterclockwise when viewed from above.

   (c) How are the above two integrals related? Explain you answer. (Are they equal? Should they be equal? If not, why not?)

2. In each of the following problems, a closed curve C and a vector field $\vec{F}$ are given. Evaluate $\oint_C \vec{F} \cdot d\vec{r}$.

   (a) C is the rectangle with vertices $(0, 0, 2)$, $(2, 0, 2)$, $(2, 2, 0)$, and $(0, 2, 0)$, in that order. The vector field $\vec{F}$ is given by $\vec{F}(x, y, z) = -4z\,\vec{i} + y^2\,\vec{j} + 3x\,\vec{k}$.

   (b) C is the curve obtained by intersecting the cylinder $x^2 + y^2 = 4$ with the surface $z = 9 - x^2$, oriented counterclockwise when viewed from above. The vector field $\vec{F}$ is given by $\vec{F}(x, y, z) = z\,\vec{i} - x\,\vec{j} + y\,\vec{k}$.

3. In each of the following problems, a surface S and a vector field $\vec{F}$ are given. Evaluate $\iint_S \vec{F} \cdot d\vec{A}$.

   (a) $\vec{F}(x, y, z) = x^2 z^3\,\vec{i} + 2xyz^3\,\vec{j} + xz^4\,\vec{k}$, and S is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$.

   (b) $\vec{F}(x, y, z) = xy \sin z\,\vec{i} + \cos(xz)\,\vec{j} + y \cos z\,\vec{k}$, and S is ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. [Hint: symmetry?]

   (c) $\vec{F}(x, y, z) = (\cos z + xy^2)\,\vec{i} + xe^{-z}\,\vec{j} + (\sin y + x^2 z)\,\vec{k}$ and S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

4. Let S be the part of the surface $z = x^2 + y$ that lies over the square in the xy-plane defined by the inequalities $-1 \leq x \leq 1$ and $0 \leq y \leq 2$, and let C be the boundary of S, oriented counterclockwise when viewed from above. Evaluate the circulation of the vector field $\vec{F}(x, y, z) = z\,\vec{i} - 2x\,\vec{j} + y\,\vec{k}$ around C.

5. Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

   (a) If $\vec{F}$ is a vector field, then div $\vec{F}$ is a vector field.

   (b) If $\vec{F}$ is a vector field, then curl $\vec{F}$ is a vector field.

   (c) If S is the sphere of radius 1, centered at the origin, oriented outward, and $\vec{F}$ is a vector field such that $\iint_S \vec{F} \cdot d\vec{A} = 0$, then div $\vec{F} = 0$ everywhere on and inside of S.

   (d) Let $\vec{F}$ and $\vec{G}$ be smooth vector fields, then

$$\text{curl}(\vec{F} \cdot \vec{G}) = \text{curl } \vec{F} \cdot \text{curl } \vec{G}$$

   (e) Let $\vec{F}$ and $\vec{G}$ be smooth vector fields, then

$$\text{curl}(\vec{F} \times \vec{G}) = (\text{curl } \vec{F}) \times (\text{curl } \vec{G})$$

6. In each of the below verify that Stokes' theorem is true for the given vector field $\vec{F}$ and surface S. This of course means to compute both

$$\iint_S \text{curl } \vec{F} \cdot d\vec{A} \qquad \text{and} \qquad \int_C \vec{F} \cdot d\vec{r}$$

   and check that they are the same. For each, comment on which was easier.

   (a) $\vec{F}(x, y, z) = y^2\ \vec{i} + x\ \vec{j} + z^2\ \vec{k}$ and S is the part of the paraboloid $z = x^2/4 + y^2$ that lies below $z = 1$, oriented upward. [Hint: the area of an ellipse is $ab\pi$, where a and b are half the lenfth of the major and minor axes.]

   (b) $\vec{F}(x, y, z) = e^{-x}\ \vec{i} + e^x\ \vec{j} + e^z\ \vec{k}$ and S is the part of the plane $2x + y + 2z = 2$ in the first octant, oriented upward.