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Spectral Estimation and Iterated Whitening

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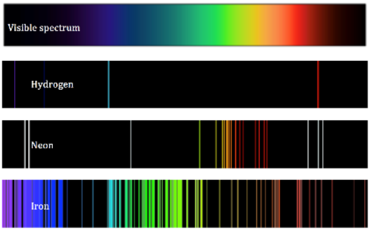
Introduction

Power spectrum: "power" or "energy" as function of frequency



Applications

- Time series analysis
- Prediction
- *Data driven model reduction*

Motivation

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Kuramoto-Sivishinsky (KS) equation

- spatiotemporal chaos

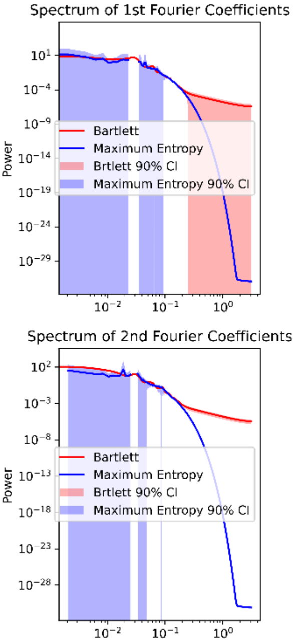
$$\begin{cases} u_t + uu_x + u_{xx} + u_{xxxx} = 0 \\ t \in [0, \infty), \quad x \in [0, L] \end{cases}$$

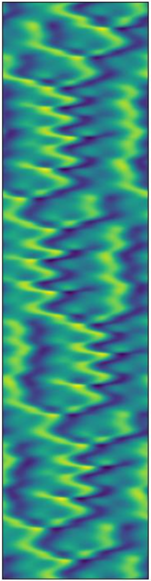
$$u(x,t) = \sum_{k=-\infty}^{\infty} v_k(t) e^{ikt}$$

$$\dot{v}_k = (q_k^2 - q_k^4) v_k - \frac{iq_k}{2} \sum_{j=-\infty}^{\infty} v_j v_{k-j}$$

$$q_k = \frac{2\pi}{L} k$$

- Our work requires *accurate spectral estimates*





What is power spectrum?

Terminology and Definition

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Given a discrete-time stochastic process (assume all necessary moments exist)

$$X = (\dots, X_{-2}, X_{-1}, X_0, X_1, X_2, \dots)$$

We have

$$\mu_n = \mathbb{E}X_n$$

$$C_X(n,m) = \mathbb{E}(X_n - \mu_n)(X_m - \mu_m)^*$$

X is *wide-sense stationary* (WSS) if

$$\mu_n = \mu \qquad \text{(constant)}$$

$$C_X(n,m) = C_X(n-m) \qquad \text{(depends only on lag)}$$

Assume X is WSS and $\mu = 0$ (WLOG)

Def. The *power spectrum* of X is

$$S_X(\omega) = \sum_{n=-\infty}^{\infty} C_X(n) e^{-i\omega n} = \mathcal{F}\{C_X\}(\omega), \qquad C_X(n) = \mathbb{E}X_n X_0^*$$

What is power spectrum?

Some intuition

Spectral representation of time series

$$X_n = \int_{-\pi}^{\pi} e^{in\omega} \sqrt{S_X(\omega)}\, dW_\omega$$

By inverse Fourier

$$\mathrm{var}(X) = C_X(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(\omega) e^{i\omega 0} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(\omega) d\omega$$

$$S_X(\omega) \approx \frac{1}{N} |\hat{X}(\omega)|^2$$

Filtering: If

$$\ell = (\dots, \ell_{-1}, \ell_0, \ell_1, \dots) \quad \text{and} \quad Y_n = (\ell * X)_n = \sum_{k=-\infty}^{\infty} \ell_k X_{n-k}$$

Then

$$S_Y(\omega) = |L(\omega)|^2 S_X(\omega), \qquad L(\omega) = \sum_{k=-\infty}^{\infty} \ell_k e^{-ik\omega}$$

White noise

$$e = (\dots, e_{-1}, e_0, e_1, \dots), \qquad e_n \text{ i.i.d. } N(0, \sigma^2)$$

Note: $S_e(\omega) = \sigma^2$

AR(p) process

$$X_n = -a_1 X_{n-1} - \cdots - a_p X_{n-p} + e_n, \qquad \{e_n\} \text{ is white noise}$$

Note: $e_n = X_n + a_1 X_{n-1} + \cdots + a_p X_{n-p} = \sum_{k=0}^{p} a_k X_{n-k} \qquad (a_0 = 1)$

$$S_e(\omega) = S_{a*X}(\omega) = |A(\omega)|^2 S_X(\omega),$$

$$A(\omega) = \sum_{k=0}^{p} a_k e^{-ik\omega}$$

$$S_X(\omega) = \frac{\sigma^2}{|A(\omega)|^2}$$

Covariance Estimation

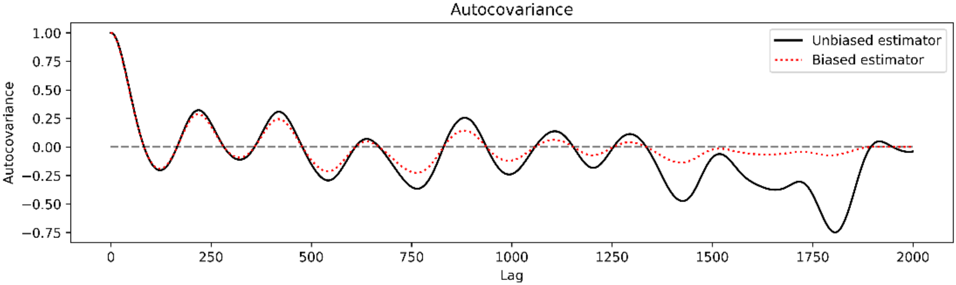
Given $x = (x_n,\ n = 1, \ldots, N)$ (a realization of $(X_n,\ n = 1, \ldots, N)$)

Unbiased Covariance Estimate:

$$R_X(n) = \mathbb{E}[(X_n - \mu)(X_0 - \mu)^*] \approx \frac{1}{N-n} \sum_{j=1}^{N-n} (x_{n+j} - \tilde{\mu})(x_j - \tilde{\mu})^*$$

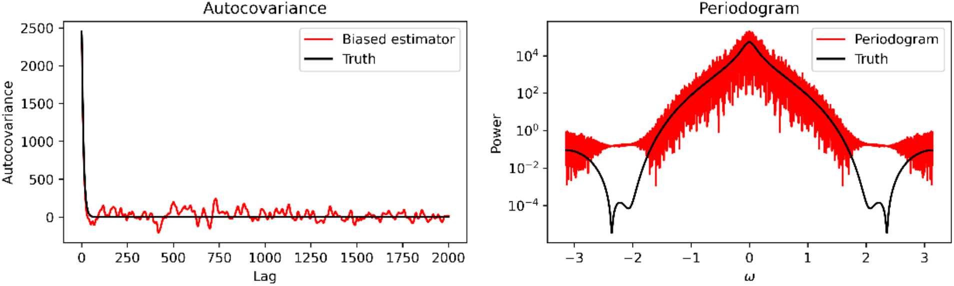
Biased estimate:

$$R_X(n) \approx \frac{1}{N} \sum_{j=1}^{N-n} (x_{n+j} - \tilde{\mu})(x_j - \tilde{\mu})^* = \tilde{R}_X(n)$$



Periodogram: (direct approach)

$$\tilde{S}_X^{\text{per}}(\omega) = \sum_n \tilde{R}_X(n) e^{-in\omega} = \cdots = \frac{1}{N} |\hat{x}(\omega)|^2 \quad (= \texttt{abs2.(fft(x))/N})$$



Periodogram: asymptotically *unbiased*, but *inconsistent*

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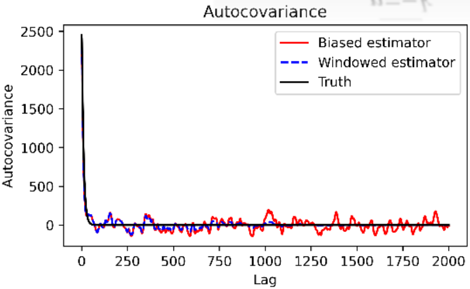
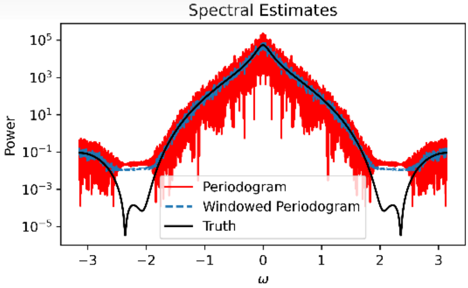
Bartlett's windowing procedure

Control *variance* but increase *bias*

$$\tilde{S}_X^{\mathrm{Par}}(\omega) = \sum_{n=-k}^{k} W^{\mathrm{Par}}(n)\tilde{R}_X(n)e^{-in\omega}$$





Periodogram: asymptotically *unbiased*, but *inconsistent*

Windowed Periodogram: asymptotically *unbiased*, and *consistent*

Maximize "entropy"

$$E = \int_{-\pi}^{\pi} \log \tilde{S}(\omega)\, d\omega$$

Subject to:

$$\int_{-\pi}^{\pi} \tilde{S}(\omega) e^{i\omega n}\, d\omega = \tilde{R}(n) \qquad \text{for } n = 0, 1, \ldots, p$$

Gives

$$\tilde{S}(\omega) = \frac{\sigma_p^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-ik\omega}\right|^2}$$

Here, $\{a_k\}_{k=1}^{p}$ are the autoregressive coefficients of order p, i.e., they minimize

$$\sum_{n=1}^{N-p} \left| x_{n+1} - \sum_{k=0}^{p-1} a_k x_{n-k} \right|^2$$

AR(p) process:

$$X_n = -a_1 X_{n-1} + \cdots - a_p X_{n-p} + e_n, \qquad e_n \text{ i.i.d. } N(0, \sigma_p^2)$$

To fit coefficients:

- Yule-Walker equations: $\{C_X(n)\}_{n=0}^{p} \Leftrightarrow \{a_n\}_{n=1}^{p}, \sigma_p^2$
    - Levinson-Durbin
    - Burg (again)

Recursive: $p = 1, \ldots, p_{\max}$

$$\begin{matrix} a_1^1 & & & & \sigma_1^2 \\ a_1^2 & a_2^2 & & & \sigma_2^2 \\ \vdots & \vdots & \ddots & & \vdots \\ a_1^{p_{\max}} & a_2^{p_{\max}} & \cdots & a_{p_{\max}}^{p_{\max}} & \sigma_{p_{\max}}^2 \end{matrix}$$

All orders < p

Use info criterion to choose p

$$\hat{S}_X^{\text{Burg}} = \frac{\sigma_p^2}{|A(\omega)|^2}, \qquad A(\omega) = 1 + \sum_{k=1}^{p} a_{k,p} e^{-ik\omega}$$

Spectral Factorization and modeling filters

Given

$$S_X(\omega) = \sum_{k=-m}^{m} C_k e^{-ik\omega}, \qquad C_{-k} = C_k^*$$

We can write

$$S_X(\omega) = |L(\omega)|^2$$

"Spectral factorization"

- Fejér-Riesz theorem
- Levinson-Durbin
- Kalman Filter

for

$$(1) \quad L(\omega) = \sqrt{S_X(\omega)} \approx \sum_{k=-m}^{m} \ell_k e^{-ik\omega} \qquad \text{or} \qquad (2) \quad L(\omega) = \sum_{k=0}^{m} \ell_k e^{-ik\omega}$$

If e is white noise with unit variance and $Y = \ell * e$, then

$$(1) \quad S_Y(\omega) = \left| \sum_{k=-m}^{m} \ell_k e^{-ik\omega} \right|^2 S_e(\omega) \approx |L(\omega)|^2 \cdot 1 = S_X(\omega)$$

$$(2) \quad S_Y(\omega) = |L(\omega)|^2 = S_X(\omega)$$

ℓ is a modeling filter

Spectral Factorization and modeling filters

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Given L, define W

$$L^{-1}(\omega) = W(\omega) \approx \sum_{k=0}^{m} w_k e^{-ik\omega}$$

Observe,

$$|W(\omega)|^2 \approx [S_X(\omega)]^{-1}$$

So,

$$\begin{aligned} S_{w*X}(\omega) &= |W(\omega)|^2 S_X(\omega) \\ &\approx S_X(\omega)/S_X(\omega) = 1 \end{aligned}$$

w is a whitening filter

Conveniently, given Bartlett estimate

$$\tilde{S}_X^{\text{Par}}(\omega) = \sum_{n=-k}^{k} W^{\text{Par}}(n)\tilde{R}_X(n)e^{-in\omega}$$

can construct approximate *modeling* and *whitening* filters

Iterated Whitening Spectral Analysis

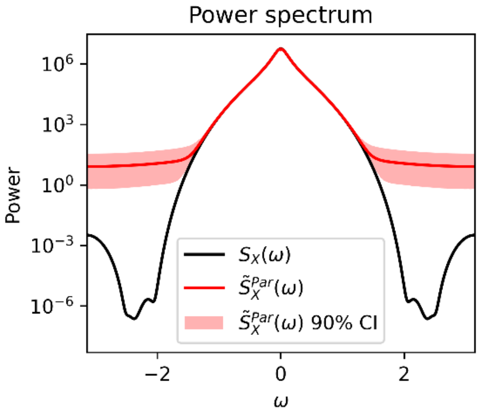
Motivating Example:

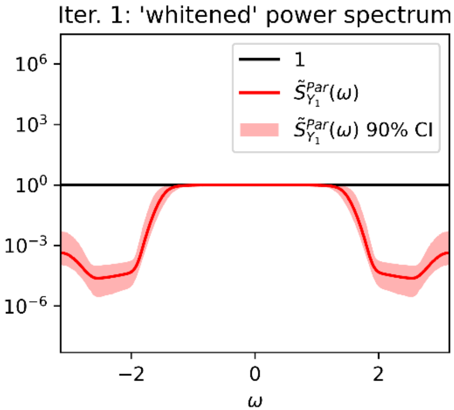
$X \sim ARMA(10,1)\ N = 10^4$,

w_1 = causal whitening filter from Parzen estimate $m = 300$

$Y_1 = w_1 * X$

Bartlett estimator has a kind of floor



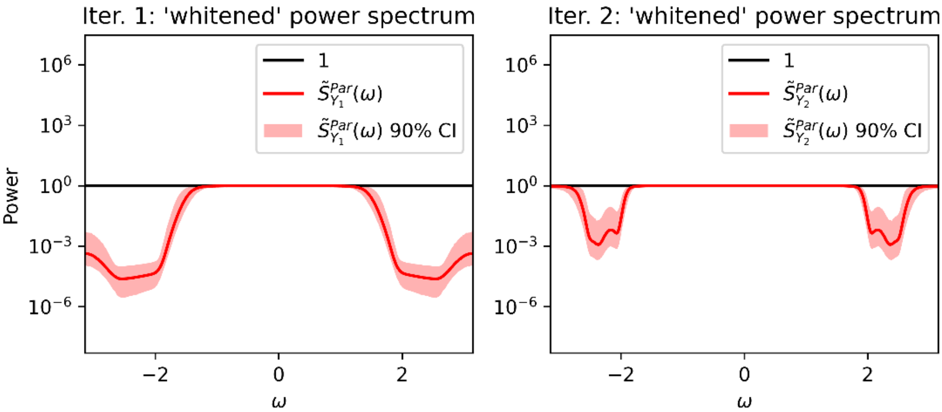


Modeling filter Spectral Analysis

What if we whiten again?

w_2 = causal whitening filter from Parzen estimate of $S_{Y_1}(\omega)$

$$Y_2 = w_2 * Y_1 = w_2 * w_1 * X$$



Modeling filter Spectral Analysis

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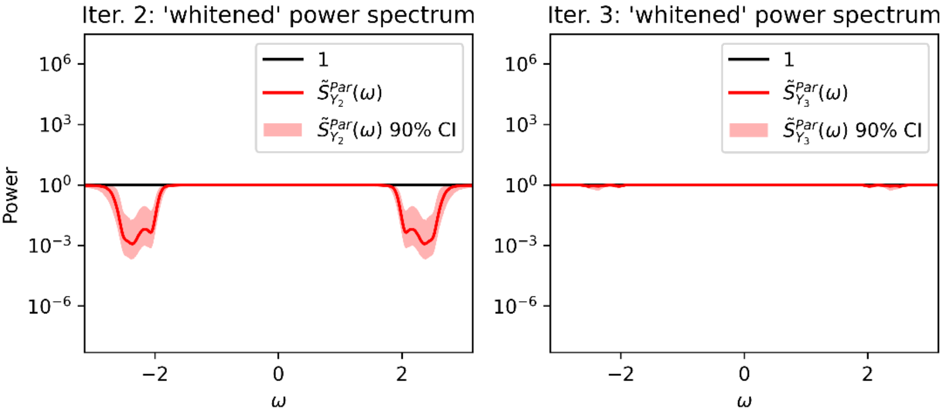
What if we whiten again?

w_3 = causal whitening filter from Parzen estimate of $S_{Y_2}(\omega)$

$Y_3 = w_3 * Y_2 = w_3 * w_2 * w_1 * X$



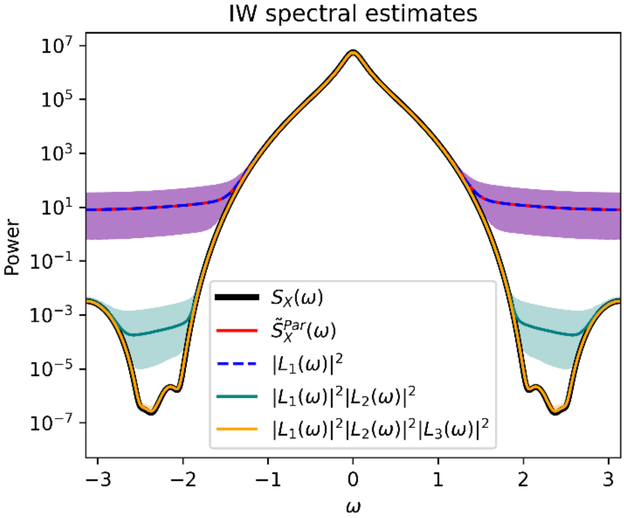
Iterated Whitening Spectral Analysis

Since $1 \approx S_{w_3 * w_2 * w_1 * X}(\omega) \approx |L_3(\omega)|^{-2} |L_2(\omega)|^{-2} |L_1(\omega)|^{-2} S_X(\omega)$

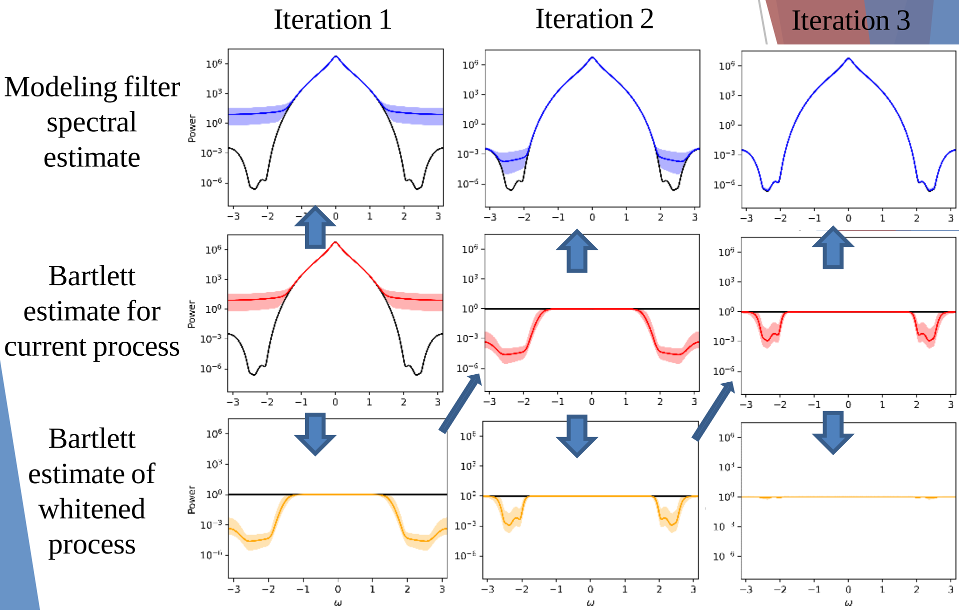
then $S_X(\omega) \approx |L_3(\omega)|^2 |L_2(\omega)|^2 |L_1(\omega)|^2 =: \tilde{S}_X^{\mathrm{IW}}(\omega)$

Collect Modeling filters $\ell^{(1)}$, $\ell^{(2)}$, and $\ell^{(3)}$ here

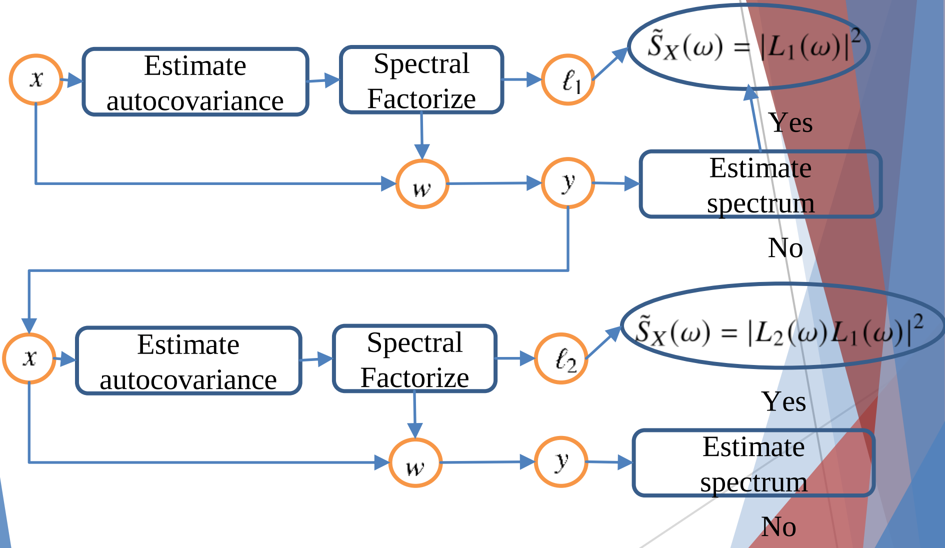
$$L_j(\omega) = \sum_{k=0}^{m} \ell_k^{(j)} e^{-ik\omega}$$



Algorithm


Iteration 1
Iteration 2
Iteration 3
Modeling filter spectral estimate
Bartlett estimate for current process
Bartlett estimate of whitened process
Power
ω

Algorithm


x
Estimate autocovariance
Spectral Factorize
ℓ_1
$\tilde{S}_X(\omega) = |L_1(\omega)|^2$
Yes
w
y
Estimate spectrum
No
x
Estimate autocovariance
Spectral Factorize
ℓ_2
$\tilde{S}_X(\omega) = |L_2(\omega)L_1(\omega)|^2$
Yes
w
y
Estimate spectrum
No

Iterated whitening converges in finite steps

- Bartlett captures top ~6 decades

- Whitening *whitens* accurate frequency range
  - New range is ~ 6 decades less

- Floor goes below 1

$$1 > \tilde{S}_X^{\mathrm{Par}}(\omega) = |L(\omega)|^2 = |W(\omega)|^{-2} \quad \Rightarrow \quad |W(\omega)|^2 > 1$$

$$S_{w*X}(\omega) = |W(\omega)|^2 S_x(\omega) > S_x(\omega)$$

Whitening raises low powers

Emanuel Parzen (1957):

$$\mathrm{var}(\hat{S}_X^{\mathrm{parz}}(\omega)) \sim \frac{151}{280}\frac{L}{N} S_X^2(\omega) \qquad (\omega \neq 0, \pm\pi)$$

$$\mathrm{bias}(\hat{S}_X^{\mathrm{parz}}(\omega)) \sim \frac{6}{L^2} S_X''(\omega)$$

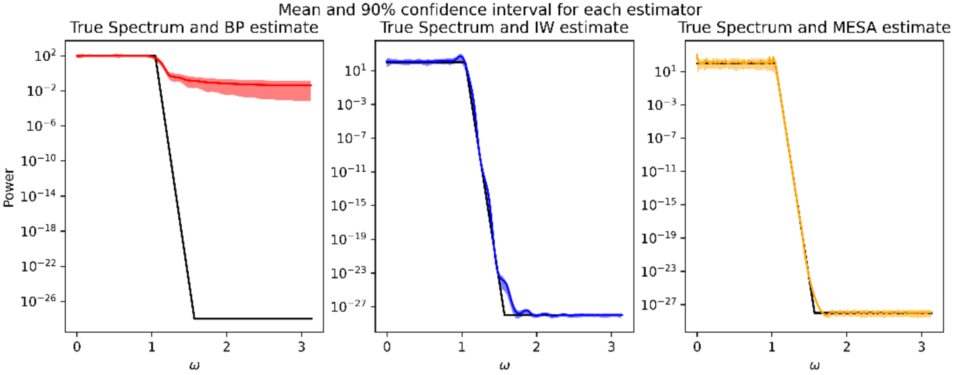
Results

Experiment

- Process know spectrum
- Large dynamic range
- N = 1000 (time series length)
- 100 independent time series

Three methods

- Bartlett-Parzen
- MESA
  - p_{max}=100
- Iterated Whitening
  - 10 iterations



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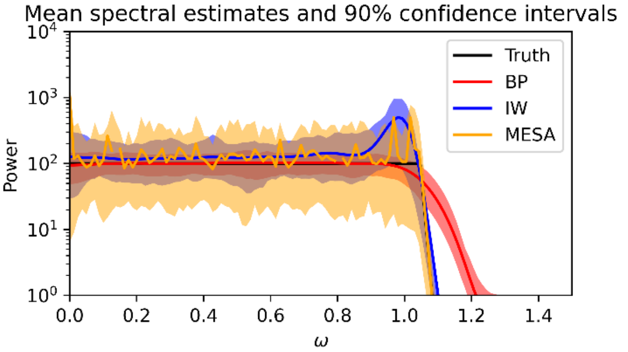
- Both MESA, IW approximate full dynamic range
- MESA has numerical issues at top of large spectral ranges
- Variance of IW less than MESA top of range, smoother



Average of 100 sample times series

Method	Spec. 1 high-low	Spec. 2 low-high-low	Spec. 3 low-high
Bartlett-Parzen	$8.55 \cdot 10^{-4}$ sec	$7.82 \cdot 10^{-4}$ sec	$7.02 \cdot 10^{-4}$ sec
Iterated Whitening	0.114 sec	0.114 sec	0.114 sec
MESA	0.00207 sec	0.00171 sec	0.00189 sec

Bartlett: $\boxed{O(mNd^2)} + O(d^2 N_f \log N_f)$

MESA: $\boxed{O(p_{\max}^2 N d^2)} + O(p_{\max} N d^3) + O(p_{\max}^2 d^3)$

IW: $\boxed{O(MmNd^2)} + O(MmN_{\mathrm{SF}} d^3)$

$$+O(Md^2 N_f \log N_f) + O(Md^3 N_f)$$

- d dimension
- M # of whitening iterations

- m lag window cutoff
- $p_{\max}$ # of Burg iterations in MESA

- N_{SF} # iterations of spectral factorization algorithm
- N_f # of points in frequency grid
- N length of time series (in steps)

KS data

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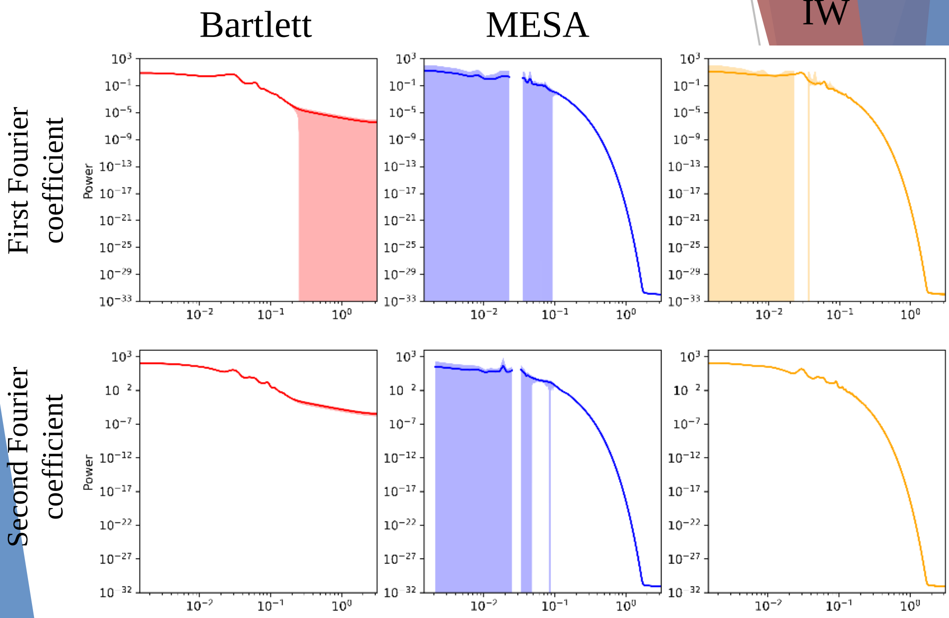

Conclusions and Future Work

Conclusion

- IW is much <u>more accurate</u> than periodogram-based estimators for spectra with <u>large dynamic range</u>
- IW is more robust than MESA
- (noncausal) IW is simple to implement

More…Multichannel

- Control Variate modified spectral estimation

Future Work

- Further analysis on convergence
- Apply estimator to model reduction
- Can iterated whitening address similar difficulties in phase estimation?

Acknowledgements

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- RTG at Arizona

Thank You!

Reference

- J. M., Kevin K. Lin, *A novel method for estimation of multi-scale spectra*, in preparation.